Formal Theory, Informally

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French Perl Workshop 2006

"I need rat poison and beer to drink."

“I need [rat poison] and [beer to drink].”

“I need [rat poison and beer] to drink.”

Formal Theory, Informally

Formal

- Describe stuff using maths and logic, not English sentences
- Mathematical notation is just another language
- However, it is formally defined, unlike English
- Enables us to say exactly what we mean, without ambiguity

Formal Theory, Informally

Theory

- Theoretical work on computation appeared before the first electronic computers
- Provides us with tools to understand what we're doing
- Provides new ideas that we can use in the real world - even if we don't see the use for them right away (for example, RSA public key cryptography)

Formal Theory, Informally

Informally

- This isn't a maths lesson
- We'll look at some stuff that's come out of the theory world...
- ...see how it helps us formally define real world stuff...
- ...and see practical uses of it.

Programming Languages

Programming Languages

- There's lots of theory that I could talk about
- I'm going to focus on the theory that helps us to build and understand programming languages and the tools that support our usage of them
- First of all: how does a program go from source code to actually being executed?

The Journey Of A Program

1. The program is tokenised

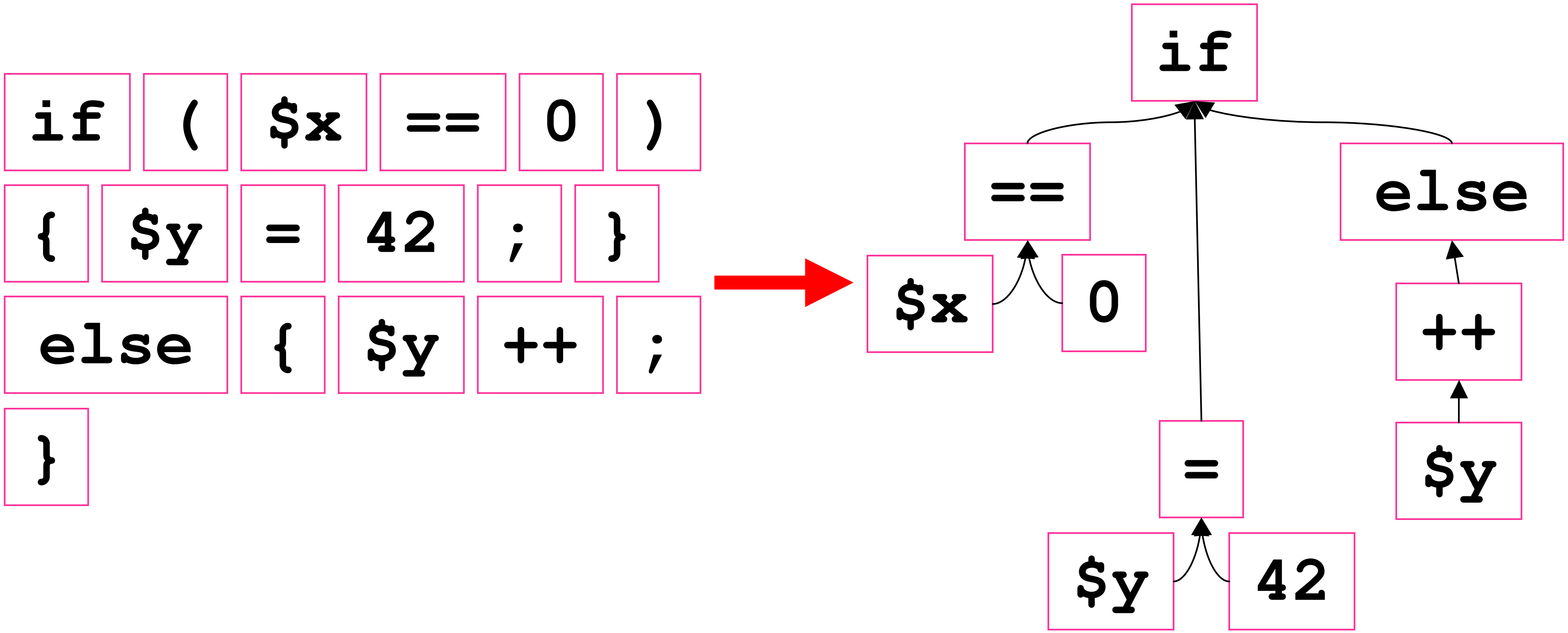
```
if ($x == 0) {
    $y = 42;
} else {
    $y++;
}
```

→

if	(	$x	==	0	)
{	$y	=	42	;	}
else	{	$y	++	;	
}					

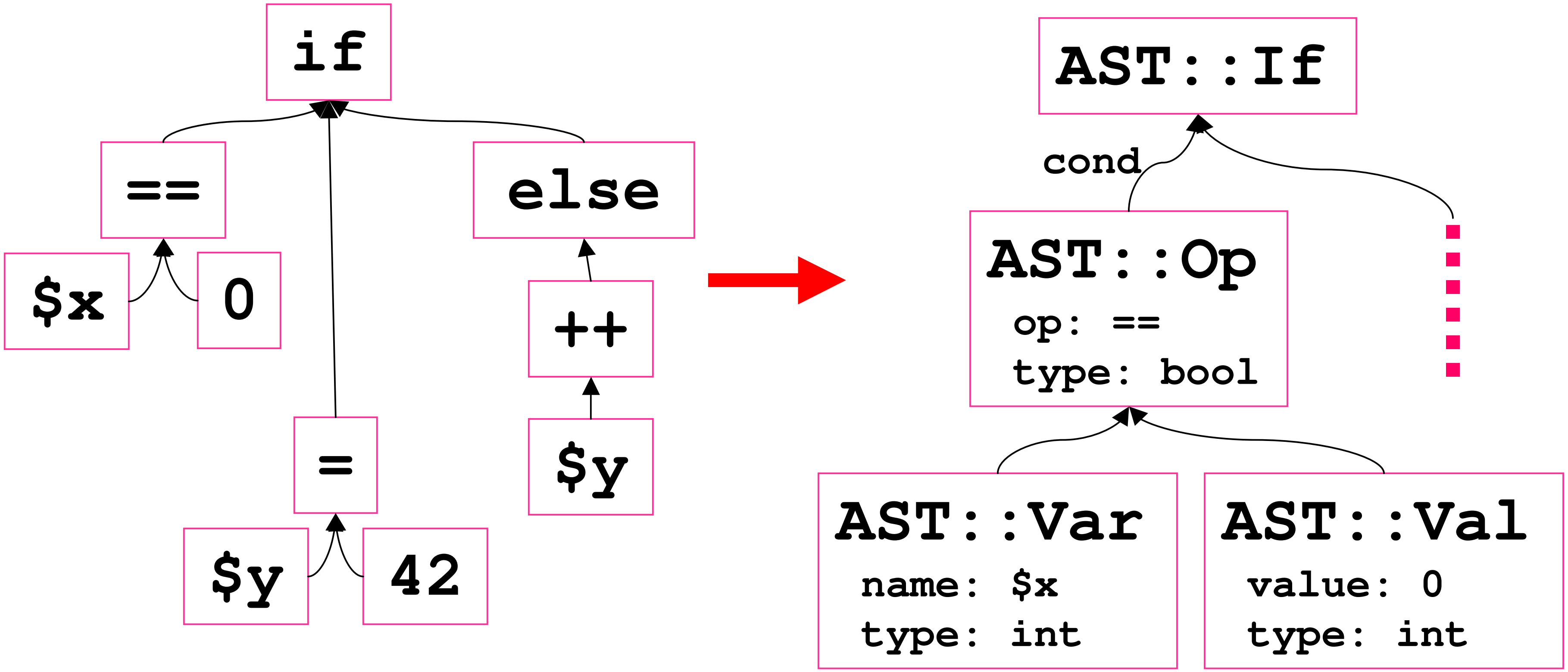
The Journey Of A Program

2. The parser takes these tokens and makes a parse tree



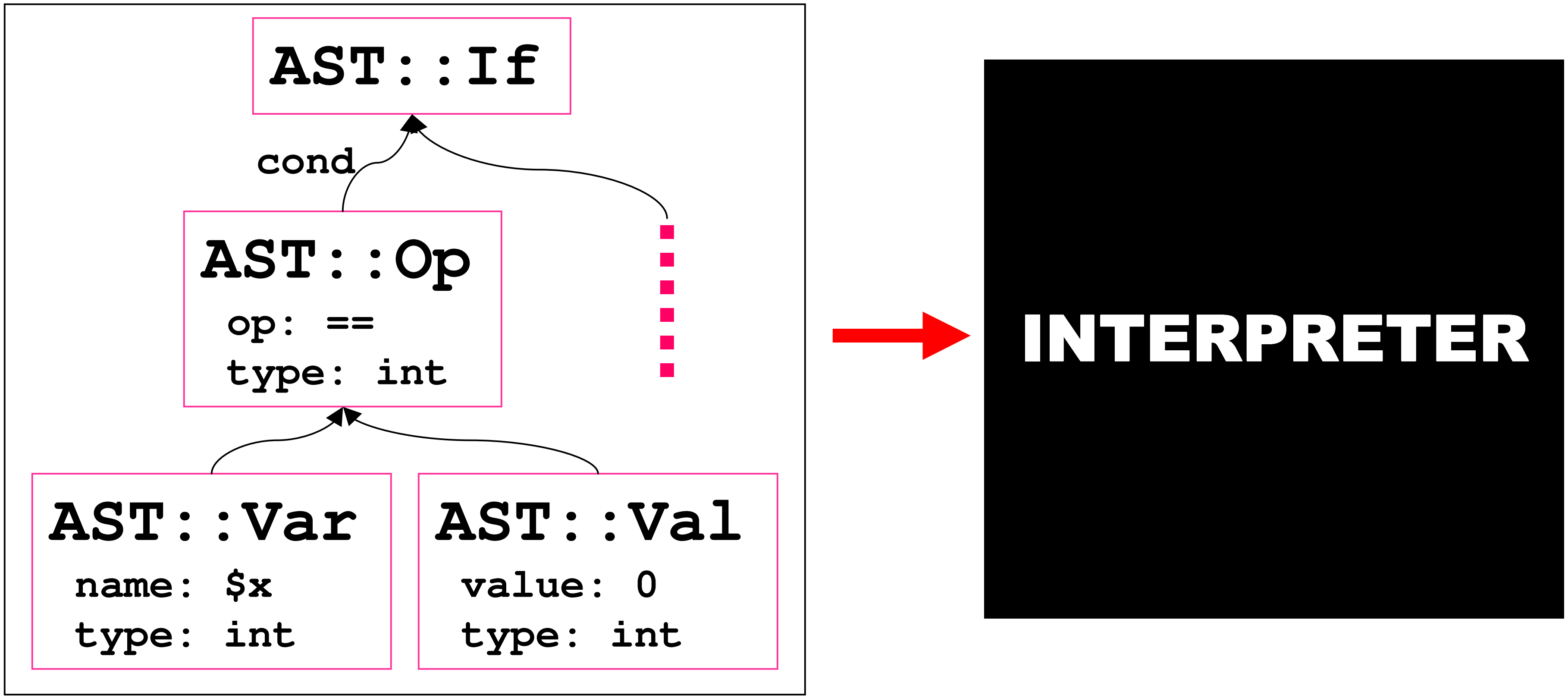
The Journey Of A Program

3. We do magical funky things to the tree and it becomes an abstract syntax tree



The Journey Of A Program

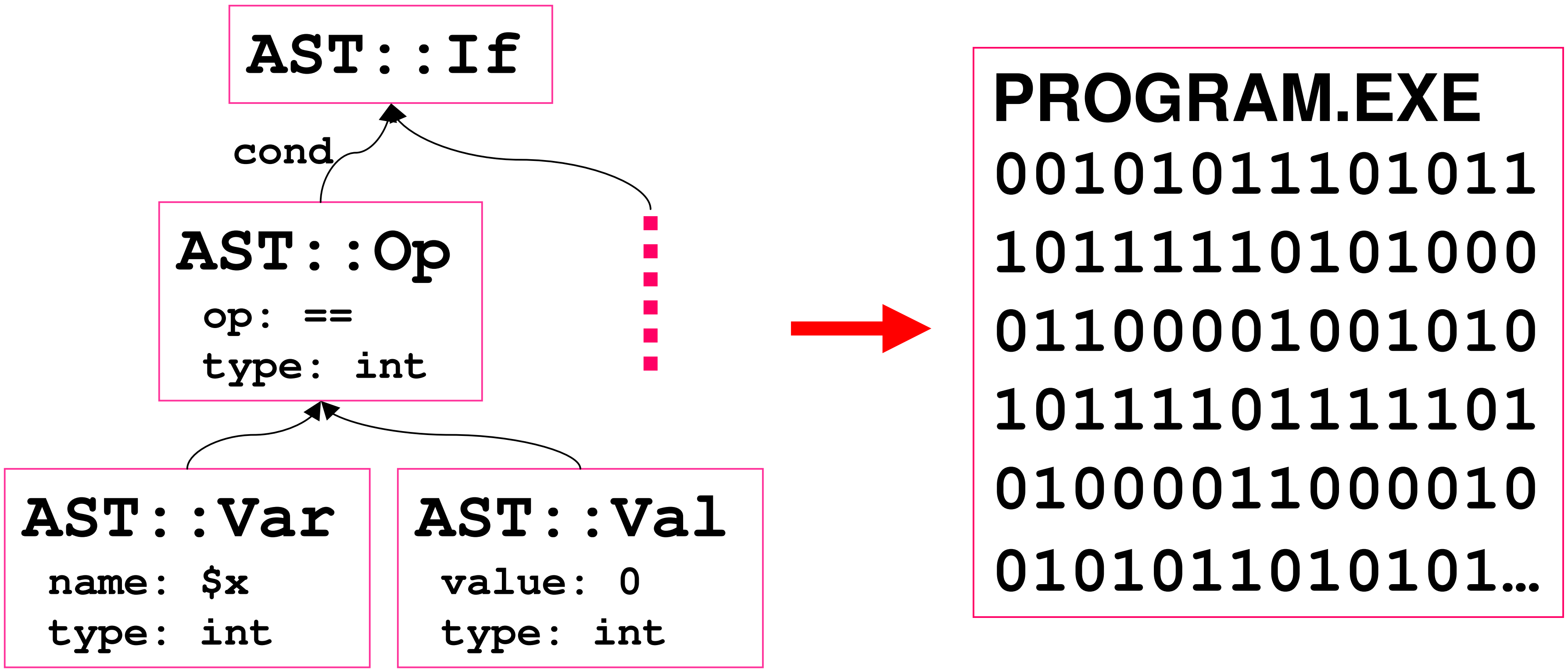
4. If we're Perl 5, we'll now walk over that tree and, for each node, do something




Alternate
Reality

The Journey Of A Program

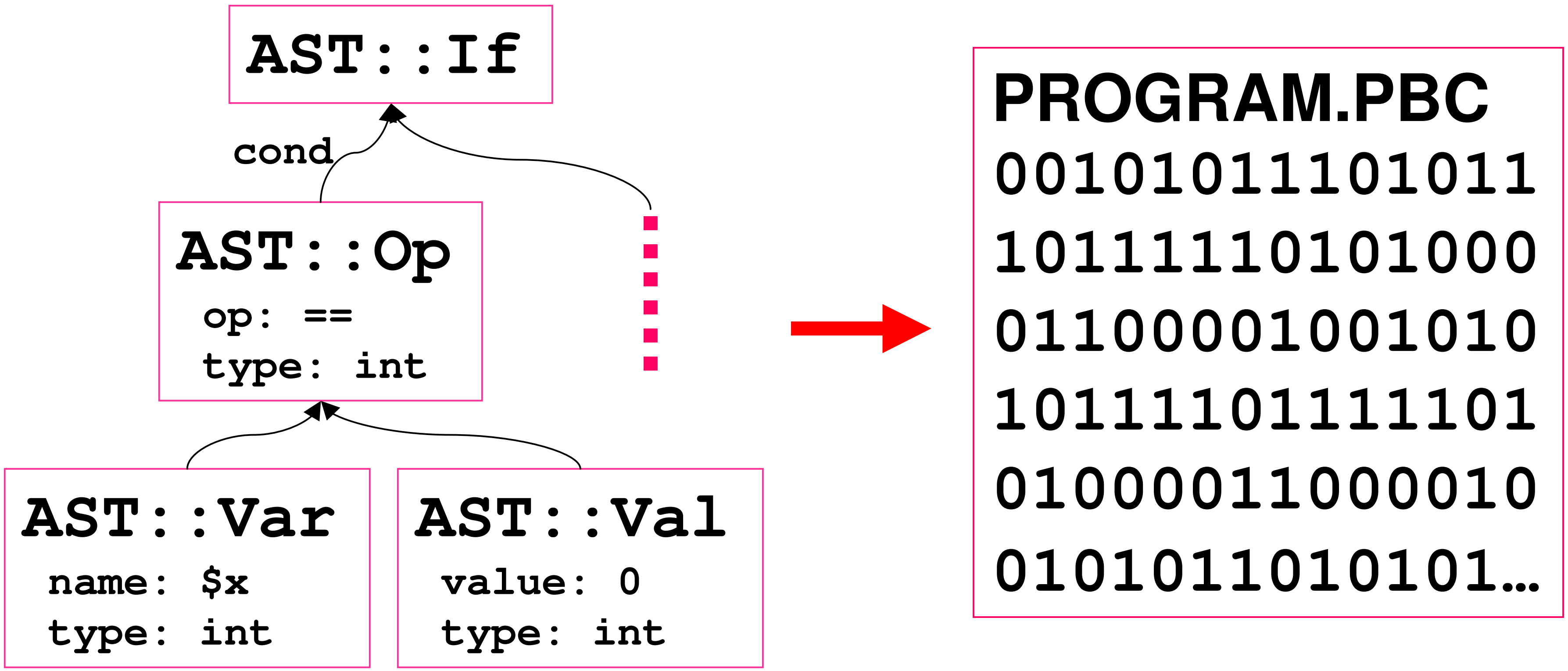
4. We walk over the tree and generate machine code for each node




Alternate
Reality

The Journey Of A Program

4. We walk over the tree and generate bytecode for a virtual machine



The Journey Of A Program

5. A virtual machine (such as the JVM or Parrot) interprets the bytecode or JIT-compiles it to machine code

PROGRAM.PBC

```
00101011101011
10111110101000
01100001001010
10111101111101
01000011000010
0101011010101...
```

Grammars

A Detour Into Linguistics

- Linguists have been analysing real languages for longer that we've had programming languages to consider
- One of the many things they came up with was the idea of a grammar
- Essentially, defining a language as a set of rules; too rigid and formal to really work for natural language, but great for programming languages!

Grammars

- Grammars are concerned with syntax, not meaning
- The grammar for a programming language can be used to generate all syntactically valid programs for that language
- **A grammar is a formal way of defining the syntax for a language**

A grammar is made up of…

- Terminals – things that we see in the language itself

```
digit ::= \d+
op ::= + | - | * | /
```

- Production rules defining non-terminals

```
expr ::= digit op expr
       | digit
```

- Note rules can be recursive (beware of what recursion is allowed – it differs)

Generation With A Grammar

- We also define a start rule: in this case, we will use **expr**.

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

- A whole program is represented by this start rule.

Parsing

- Grammars are most commonly used to parse programs rather than generate them.
  - Take a program
  - Work out what grammar rules you need to get back to the start rule from the tokens the program is made up of

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

```
35 + 7
```

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7

digit: 35

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7

digit: 35

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7

digit: 35	op: +

Formal Theory, Informally

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7

digit: 35 | op: +

Formal Theory, Informally

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

```
35 + 7
```

```
digit: 35    op: +    digit: 7
```

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7

digit: 35	op: +	digit: 7

Parsing

- Result is that we build a parse tree

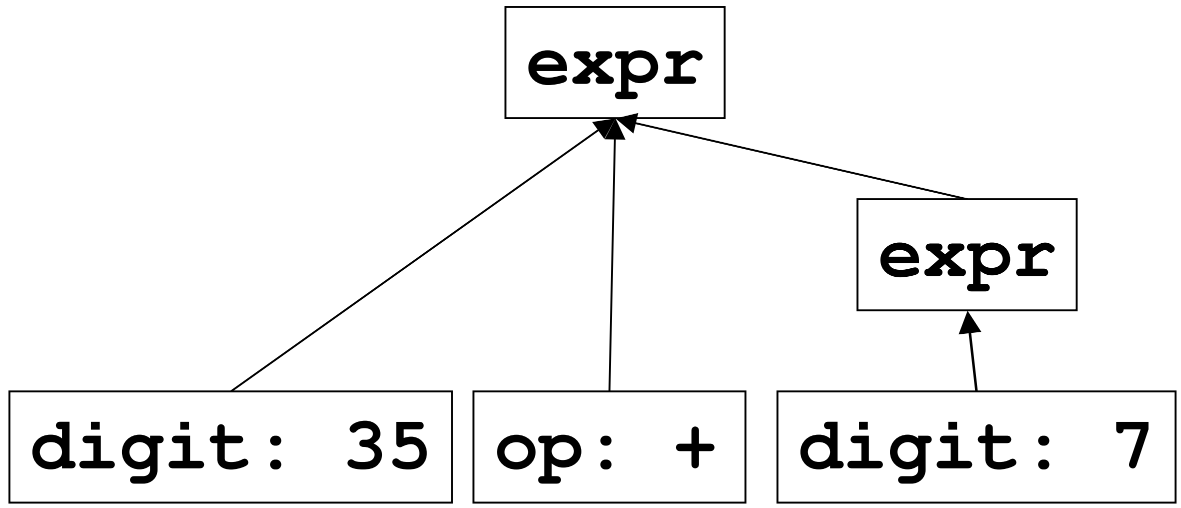
```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

Parsing

- Result is that we build a parse tree

```
expr ::= digit op expr
       | digit
digit ::= \d+
op ::= + | - | * | /
```

35 + 7



Grammars In Perl 6

- Can translate our example directly into Perl 6.

```
grammar Math {
    token op    { <'/'> | <'*'>
                | <'+'> | <'-'> }
    token digit { \d+ }
    token expr  { <digit> <op> <expr>
                | <digit> }
}

my $tree = "35+7" ~~ /^<Math.expr>$/;
```

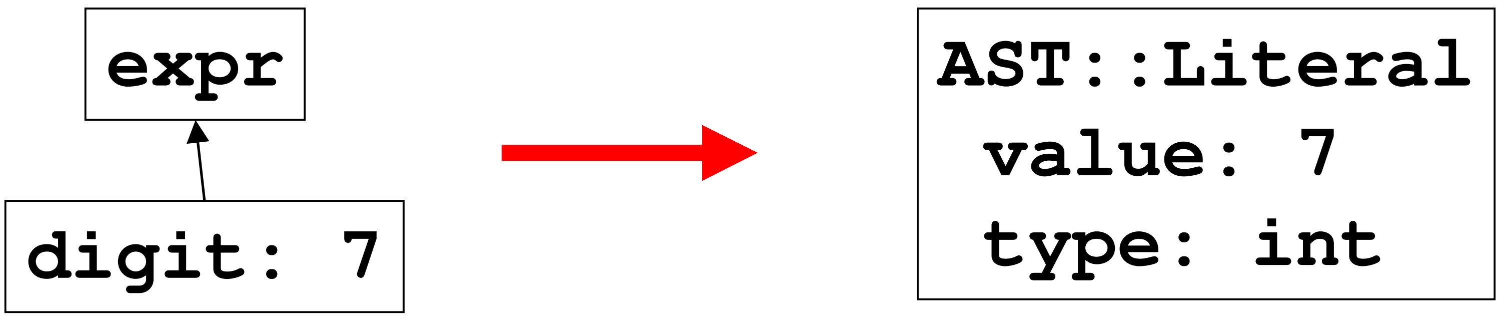
Attribute Grammars

Mostly A Scary Name

- Attribute grammars might sound less scary if we called them Tree Grammars
- They are used in the Tree Grammar Engine, part of the Parrot compiler tools
- Instead of taking a string of characters as input, tree grammars take a tree
- Specify a “transform” to perform on each type of node in the tree

Abstract Syntax Trees

- Aim is to capture the semantics, but without the mess in the parse tree that was a result of the language's syntax
- Also annotate nodes with extra stuff – perhaps types



Writing Attribute Grammar Transforms

- This is TGE-like syntax (you can’t write Perl 6 to implement the transform yet, only PIR)
- Here’s the rule for **`digit`** nodes

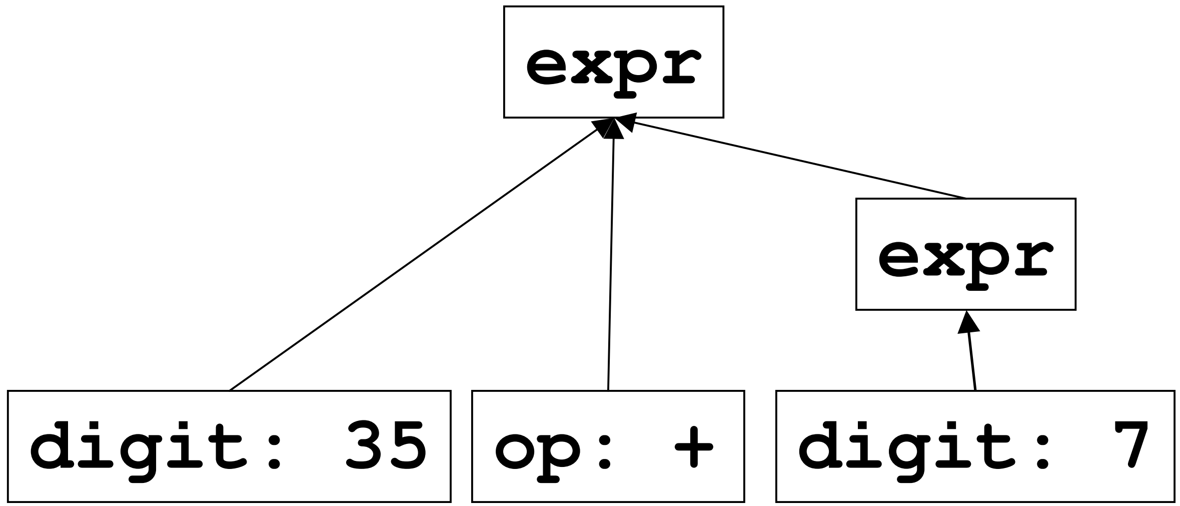
```
transform make_ast (digit) {
    $result = new AST::Literal;
    $result.value = $node;
    $result.type = 'int'
}
```

Writing Attribute Grammar Transforms

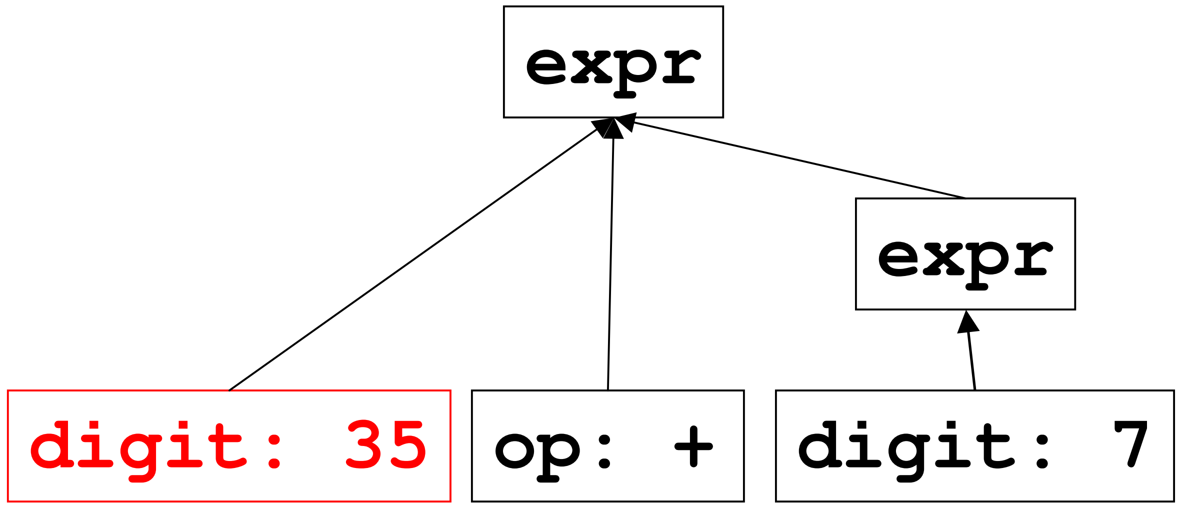
- The rule for **`expr`** is more complex

```
transform make_ast (expr) {
    if $node<op> {
        $result = new AST::Op;
        $result.opname = $node<op>;
        $result.oper1 = $node<digit>;
        $result.oper2 = $node<expr>;
    } else {
        $result = $node<digit>;
    }
}
```

From Parse Tree To AST

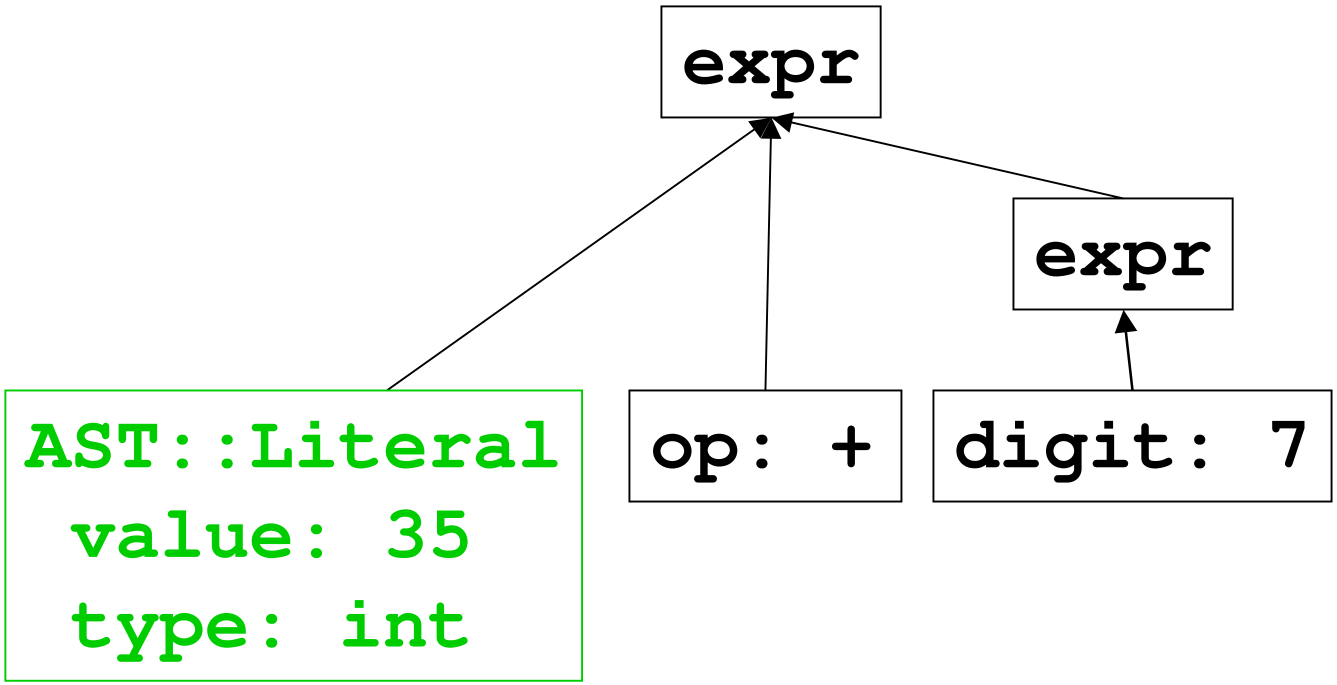


From Parse Tree To AST

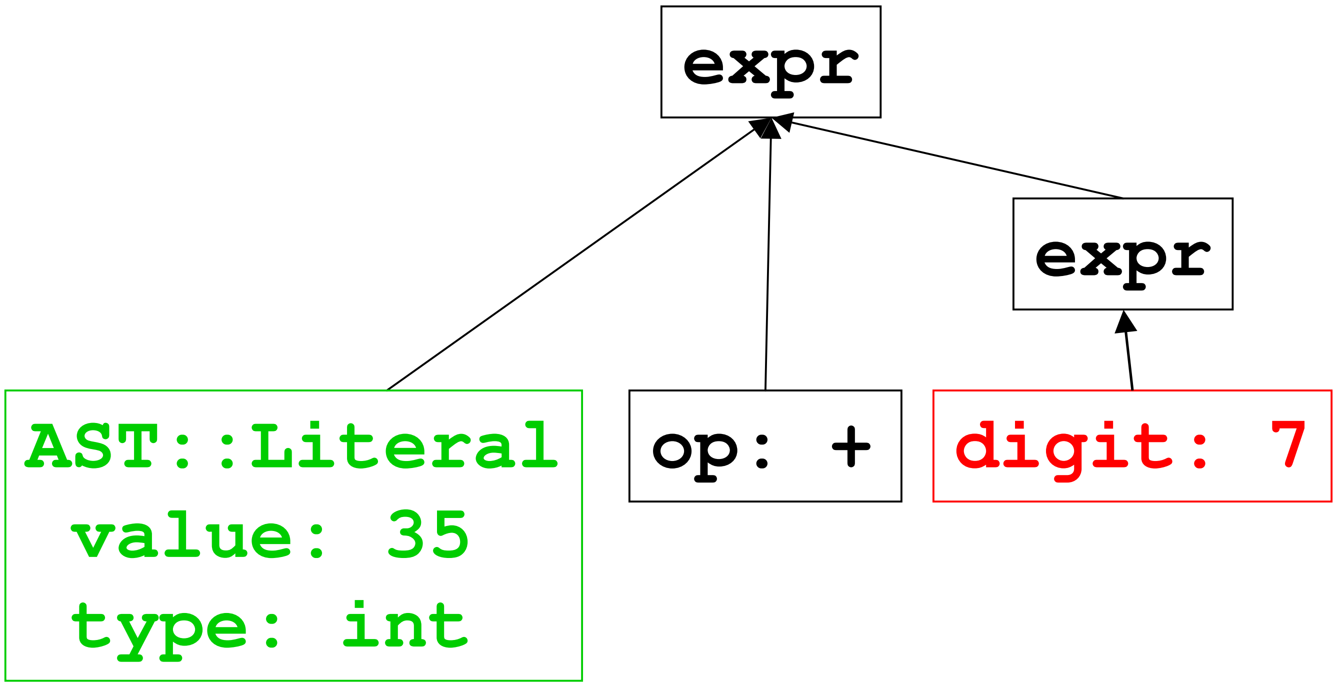



```
transform make_ast (digit)
```

From Parse Tree To AST

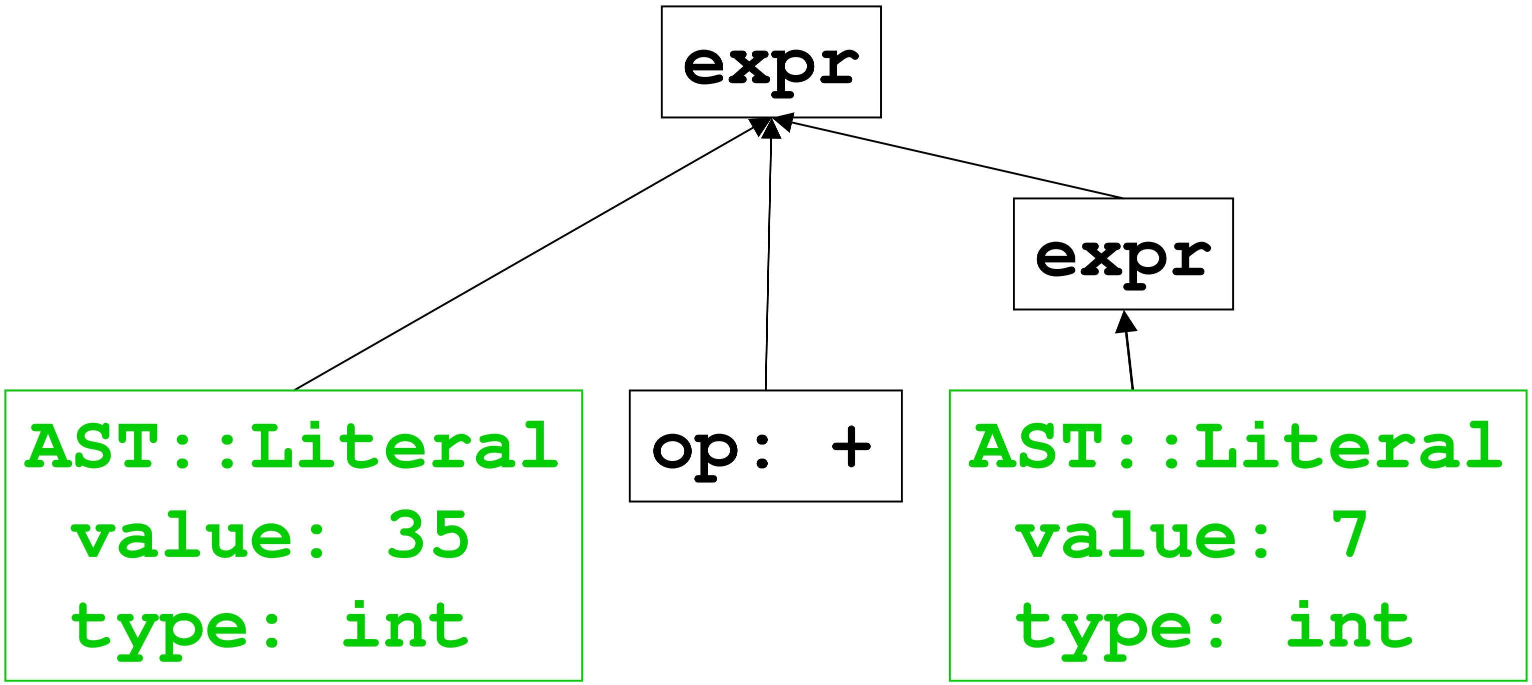


From Parse Tree To AST

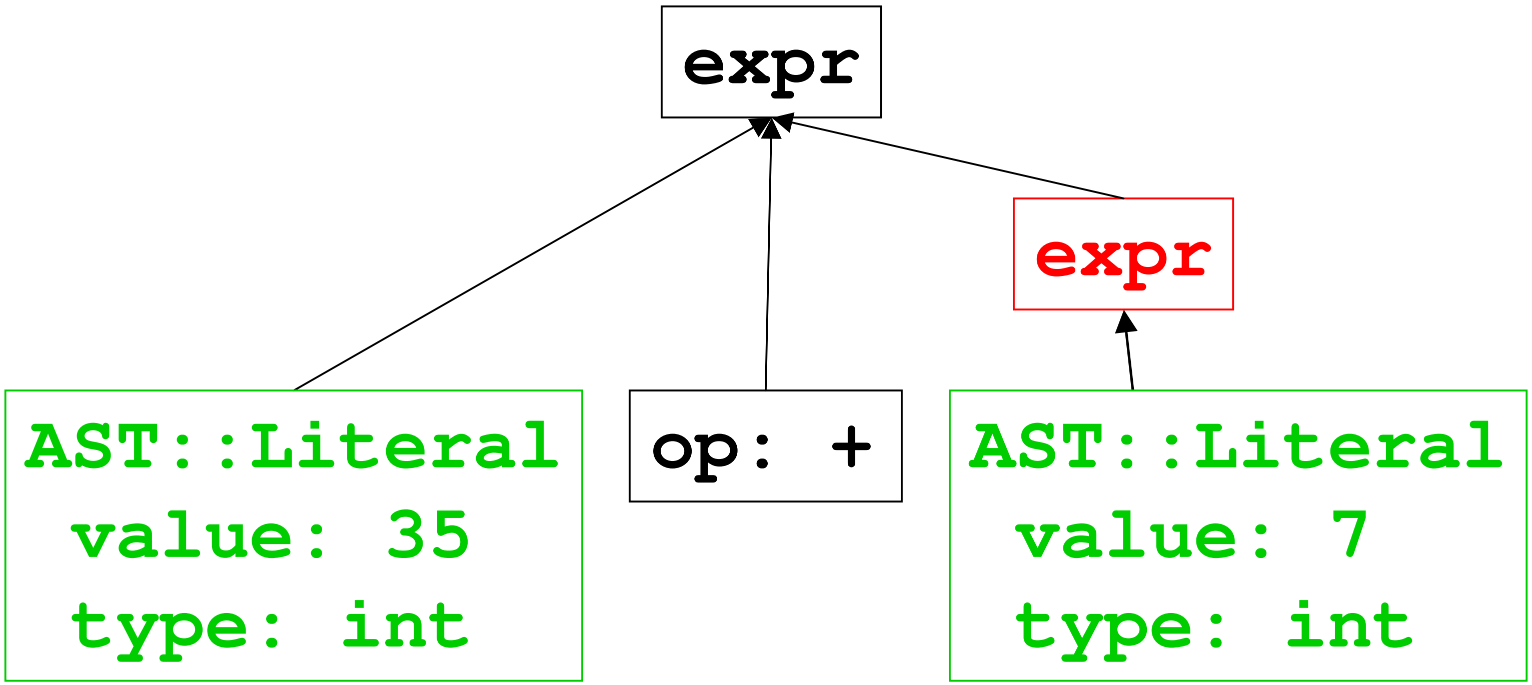


`transform make_ast (digit)`

From Parse Tree To AST

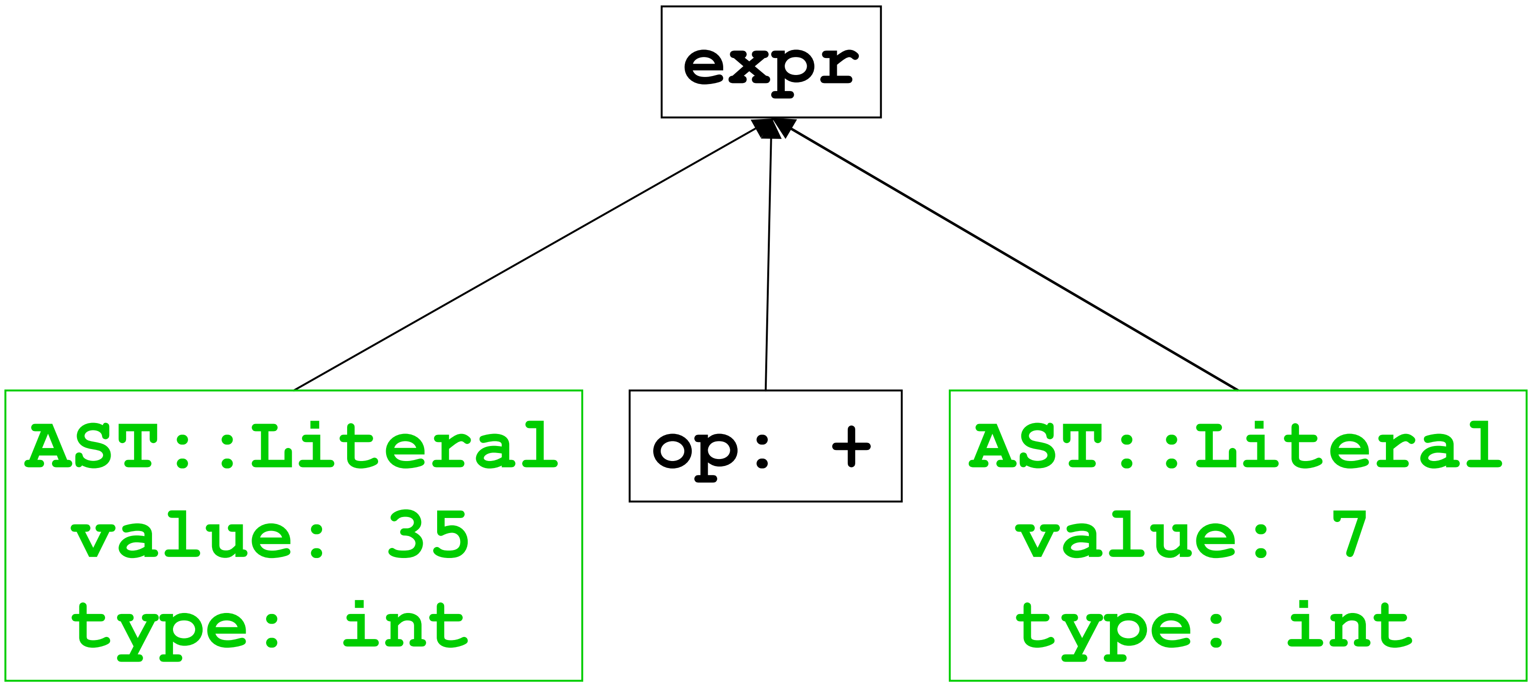


From Parse Tree To AST

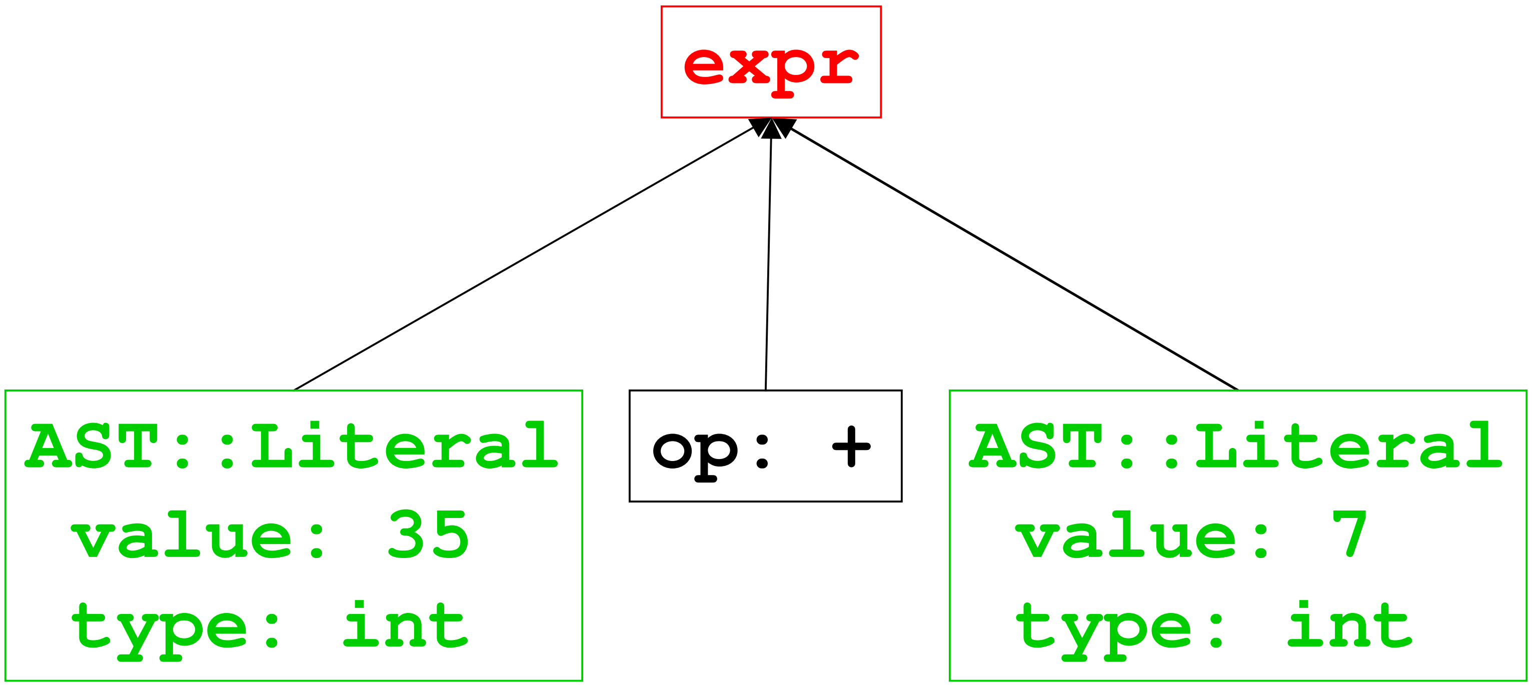



```
transform make_ast (expr)
```

From Parse Tree To AST

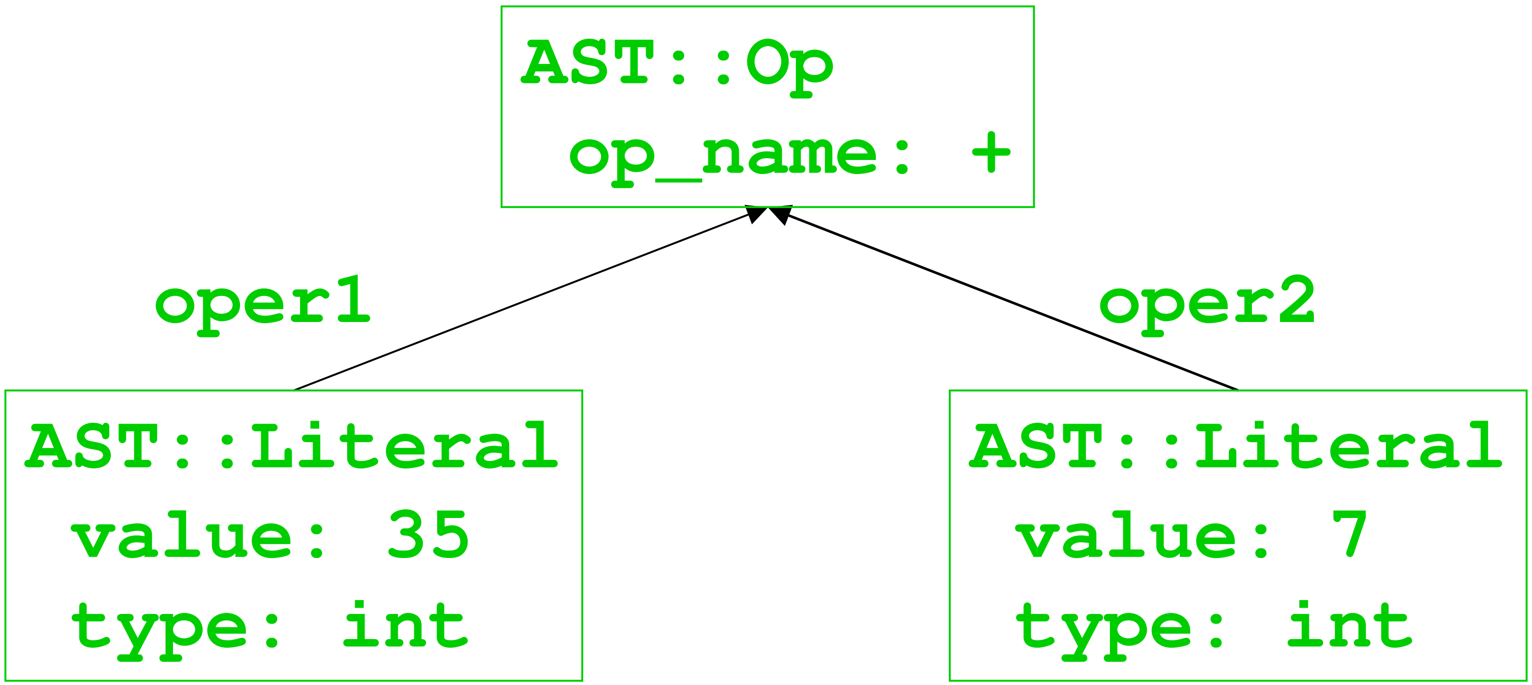


From Parse Tree To AST



`transform make_ast (expr)`

From Parse Tree To AST



Formal Semantics

Oh, behave!

- Grammars enabled us to formally specify the syntax of a language
- Formal semantics is about formally specifying the behaviour of the language

Operational Semantics

- We formalize the execution of the program by taking steps according to a sequence of evaluation rules
- These evaluation rules are what formally define the language
- In the examples I will demonstrate, at any point in the execution we will have the current term that is being evaluated and a store (mapping names to values)

Operational Semantics

- We will take a very simple language to define the semantics for
- It's helpful to see the syntax first – here it is specified as a grammar

```
value ::= n | x | true | false
    (n is an integer, x is a name)
term ::= if expr then expr else expr
       | expr + expr
       | expr == expr
expr ::= value | term
```

Inductive Evaluation Rules

- Terms in our program fall into two categories
  - Things we can evaluate right away (for example, 39 + 3) – rules for these are our **base cases**
  - Things we need to evaluate part of first (for example, (27 + 12) + 3) – rules for these are our **inductive steps**

Inductive Evaluation Rules

- The key idea behind induction: we can always break a program down until we get to base cases
- This provides us with a mechanism for proving a semantics have a property:
  - Prove it for the base cases
  - Prove that inductive steps retain the property

Evaluation Rules – Base Cases

$$\frac{}{(n_1 + n_2, s) \rightarrow (n, s)} \text{ (when } n = n_1 + n_2)$$

$$\frac{}{(n_1 == n_2, s) \rightarrow (true, s)} \text{ (when } n_1 = n_2)$$

$$\frac{}{(n_1 == n_2, s) \rightarrow (false, s)} \text{ (when } n_1 \neq n_2)$$

- s represents the store (mapping names to values)
- → represents a step of computation
- n, n_1 and n_2 represent integers

Evaluation Rules – Base Cases

$$\frac{}{(if\ true\ then\ t_1\ else\ t_2, s) \rightarrow (t_1, s)}$$

$$\frac{}{(if\ false\ then\ t_1\ else\ t_2, s) \rightarrow (t_2, s)}$$

- t_1 and t_2 represent other terms in the program
- Essentially, if the condition is true, the term as a whole reduces to the "then" cause, otherwise it reduces to the "else" clause

Evaluation Rules – Inductive Steps

$$\frac{(t_1, s) \rightarrow (t_1', s)}{(t_1 + t_2, s) \rightarrow (t_1' + t_2, s)}$$

$$\frac{(t_2, s) \rightarrow (t_2', s)}{(n_1 + t_2, s) \rightarrow (n_1 + t_2', s)}$$

- You can read the first rule as “if I have two terms added together, I do a step of evaluation on the first term”
- Note that these two rules encode that we evaluate left to right for addition!

Evaluation Rules – Inductive Steps

- The rest of the inductive steps pretty much follow this pattern
- Remember how in the grammar I carefully separated terms from values
- This means that our rules are deterministic – there is always at most one rule we can choose
- If no possible rule, the program is stuck

Example Evaluation

- Here is an example evaluation using the rules that we defined.

```
(if x == 0 then 42 else 12, {x→0})
```

Example Evaluation

- Here is an example evaluation using the rules that we defined.

```
   (if x == 0 then 42 else 12, {x→0})
→ (if 0 == 0 then 42 else 12, {x→0})
```

Example Evaluation

- Here is an example evaluation using the rules that we defined.

```
   (if x == 0 then 42 else 12, {x→0})
→ (if 0 == 0 then 42 else 12, {x→0})
→ (if true then 42 else 12, {x→0})
```

Example Evaluation

- Here is an example evaluation using the rules that we defined.

```
    (if x == 0 then 42 else 12, {x→0})
→  (if 0 == 0 then 42 else 12, {x→0})
→  (if true then 42 else 12, {x→0})
→  (42, {x→0})
```

An Evaluation That Gets Stuck

- Evaluating this will get to a state where no rules apply

```
(if x + 5 then 42 else 12, {x→3})
```

An Evaluation That Gets Stuck

- Evaluating this will get to a state where no rules apply

```
    (if x + 5 then 42 else 12, {x→3})
→ (if 3 + 5 then 42 else 12, {x→0})
```

An Evaluation That Gets Stuck

- Evaluating this will get to a state where no rules apply

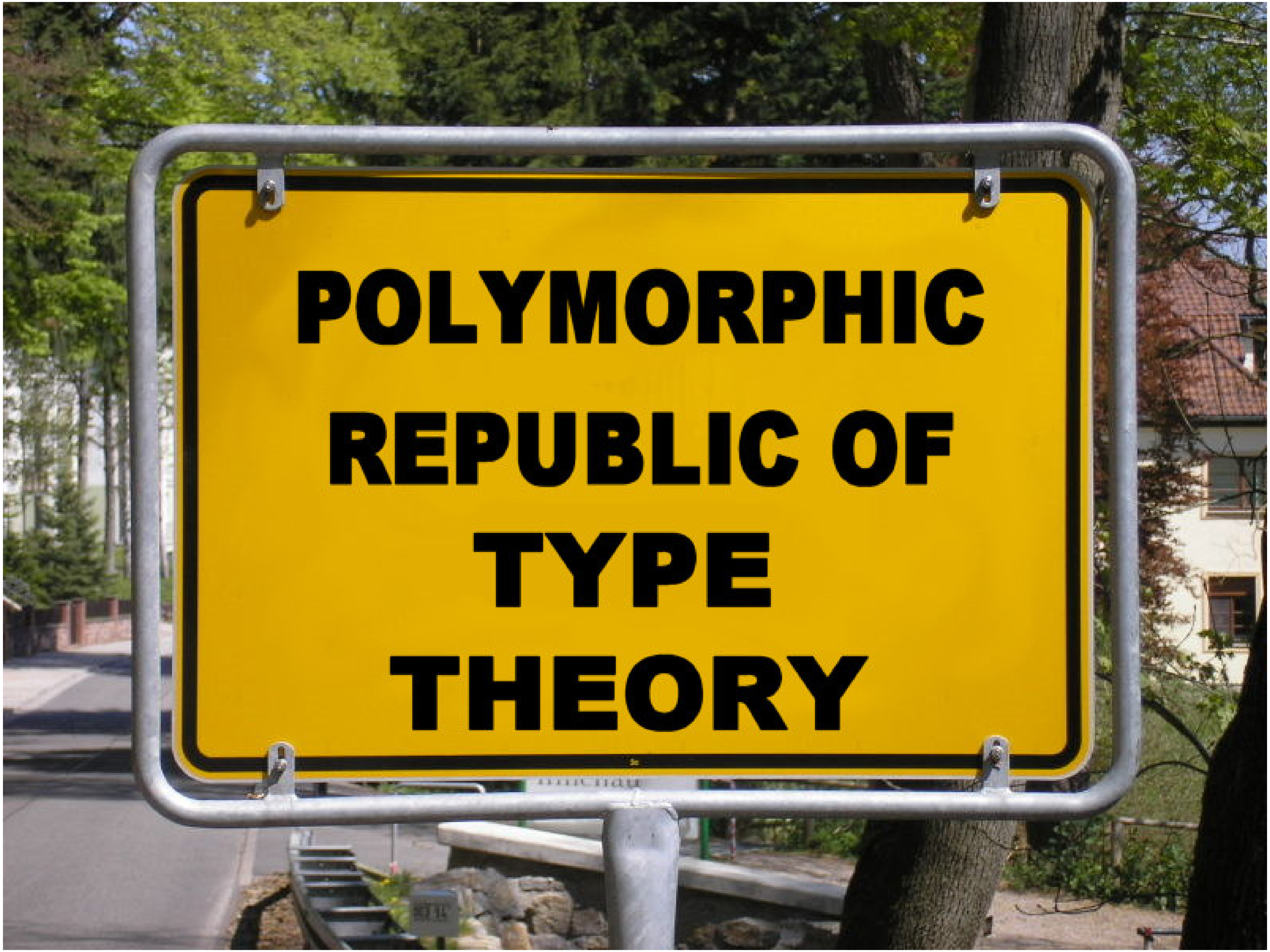
```
   (if x + 5 then 42 else 12, {x→3})
→ (if 3 + 5 then 42 else 12, {x→0})
→ (if 8 then 42 else 12, {x→0})
```

An Evaluation That Gets Stuck

- Evaluating this will get to a state where no rules apply

```
   (if x + 5 then 42 else 12, {x→3})
→ (if 3 + 5 then 42 else 12, {x→0})
→ (if 8 then 42 else 12, {x→0})
→ FAIL
```

- Would like to turn down programs like this somehow at compile time


POLYMORPHIC
REPUBLIC OF
TYPE
THEORY

What Is A Type?

- TMTOWTDI (There's More Than One Way To Define It)
- A common definition: a type classifies a value (e.g. 42 is an integer, "monkey" is a string…)
- Another definition: a type defines the representation of and set of operations that can be performed on a value

What Is A Type System?

- Real programs consist of terms that compute values
  - “`29 + 13`”
- A type system classifies a term in a program according to the type of values that it will compute
  - “`29 + 13`” will have type “`integer`”
- Vary greatly between languages

Formalizing Types

- We usually specify that a term has a type by placing a colon between the two

$$42 : int$$

$$1 + 5 : int$$

$$true : bool$$

Type Environments

- A type environment, often written Γ (uppercase Greek letter gamma), maps names (of variables in languages that have them) to types
- For example, the following type environment tells us the types of the scalars $x and $b

$$\Gamma = \{\ \$x \rightarrow\ int,\ \$b \rightarrow\ bool\ \}$$

Type Environments

- The type environment Γ on the last slide allows us to determine the following typing:

$$\boxed{2 * \$x : int}$$

- Formally we write this as follows:

$$\boxed{\Gamma \vdash 2 * \$x : int}$$

- Which we read as “gamma proves that 2 * $x has type int”

Inductive Typing Rules

- We use inductive rules, just like we did with operational semantics
- Here are some the base cases for our type system – the types for values

$$\overline{\Gamma \vdash n : int} \text{ (provided } n \text{ is an integer)}$$

$$\overline{\Gamma \vdash true : bool}$$

$$\overline{\Gamma \vdash false : bool}$$

$$\overline{\Gamma \vdash x : T} \text{ (provided } \Gamma(x) = T\text{)}$$

Inductive Typing Rules

- Addition could have this typing rule:

$$\frac{\Gamma \vdash t_1 : int \qquad \Gamma \vdash t_2 : int}{\Gamma \vdash t_1 + t_2 : int}$$

- You can read this as "we can prove that $t_1 + t_2$ has type int provided that t_1 has type int and t_2 has type int"
- The conditions above the line must be true for the what is below the line to be

Inductive Typing Rules

- The typing rule for “if” is a little more complex; we introduce a type variable T:

$$\frac{\Gamma \vdash t_1 : bool \quad \Gamma \vdash t_2 : T \quad \Gamma \vdash t_3 : T}{\Gamma \vdash if\ t_1\ then\ t_2\ else\ t_3 : T}$$

- This specifies that the condition of the if statement must be a boolean and the branches of the if must have the same type (not true of all languages!)

Type Checking

- Given a type environment, a term and the type that we believe the term to have, type checking verifies that the term does indeed have that type

Given a type environment Γ, a term t and a type T, show that $\Gamma \vdash t : T$

- By doing type checking at compile time with the typing rule for "if" shown on the last slide, our stuck example from earlier is now rejected at compile time!

Polymorphism

- Again, TMTOWTDI (both for D = Define and D = Do)
- One definition: polymorphism occurs when a term or value can be classified as having more than one type
- Another definition: polymorphism allows the same code to operate on values of different types

Polymorphism

- Many ways to achieve polymorphism
  - Subclassing
  - Parametric polymorphism (aka generics and parameterised types)
  - Refinement types
- We can formalize all of these
- Will just look at how we formalize subclassing and refinement types

Subclassing (Inheritance)

- Perl 6 has some nicer syntax for defining a subclass than Perl 5:

```
class Melon is Fruit {
    ...
}
```

- We formalize subclassing by adding a sub-typing rule that looks something like this (we really need to define "isa")

$$\frac{\Gamma \vdash t : S \qquad S \; isa \; T}{\Gamma \vdash t : T}$$

Refinement Types

- A refinement type is obtained by adding constraints to an existing type
- For example, the type EvenInt is a refinement of the Int type that only contains even integers
- In Perl 6, EvenInt would be defined like this:

```
subset EvenInt of Int where { $^n % 2 == 0 }
```

Refinement Types

- Can use a more refined type in place of a less refined one (e.g. EvenInt in place of Int)
- We formalize this using the denotation of the type, which is basically the set of all values of that type.

$$\frac{\Gamma \vdash t : S \qquad [\![S]\!] \subseteq [\![T]\!]}{\Gamma \vdash t : T}$$

The End

Any questions?